8. Research directions in polytope theory

28/11/22

OVERVIEW ... beyond the lecture

geometry / topology

- rigidity theory
- spectral polytope theory
- symmetric polytopes
- inscribed polytopes

combinatorics

- zonotopes & hyperplane arrangements
- triangulations & dissections
- Stanley-Reisner rings & Cohen-Macauley complexes
- polytope algebra

algebra

- toric varieties
- lattice polytopes

algorithms / complexity

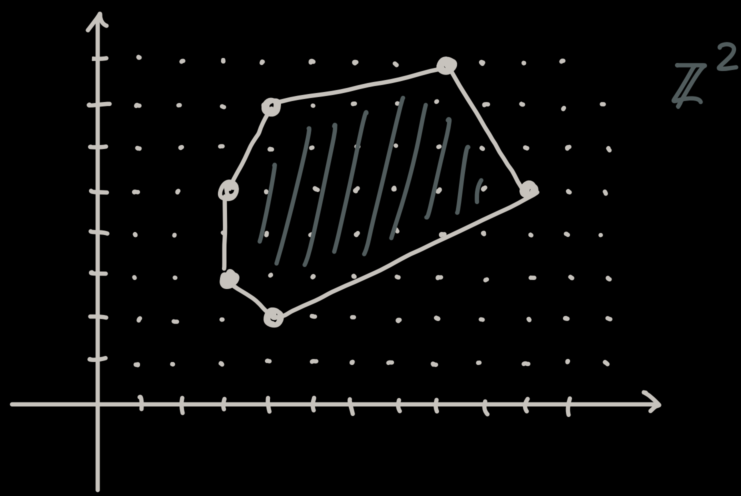
- computational polytope theory
- simplex algorithm and the Hirsch conjecture
- abstract polytopes

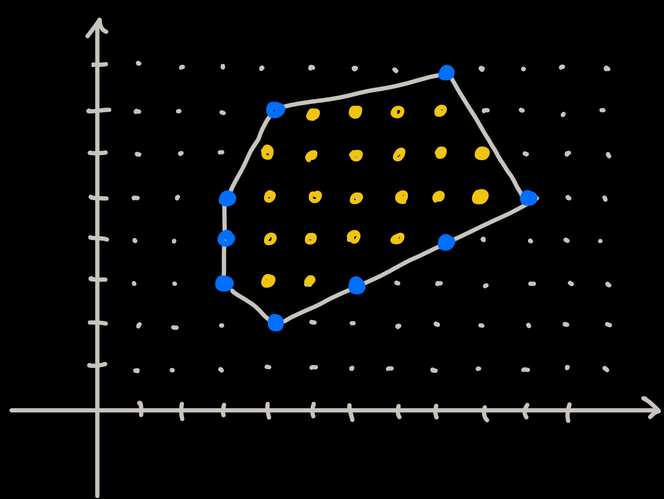
generalizations

- directed polytopes
- complex / quaternionic polytopes
- spherical / hyperbolic polytopes
- polyhedra = unbounded polytopes
- non-convex polytopes
- infinite-dimensional polytopes
- convex bodies

8.1 Lattice polytopes

Def: a lattice polytope has all its vertices in $\mathbb{Z}^d$.

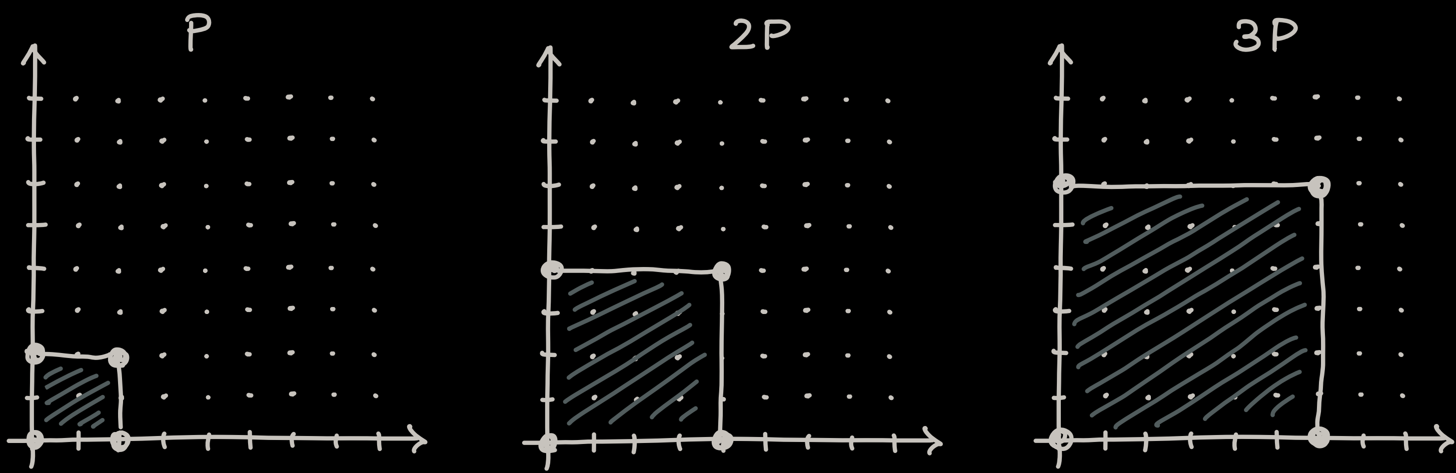




Thm: (Pick's theorem) 1899

$$\text{Area} = i + \frac{b}{2} - 1$$

Ehrhart theory



$\mathrm{vol}_L = 4$ $\mathrm{vol}_L = 16$ $\mathrm{vol}_L = 36$

$\longrightarrow \mathrm{vol}_L := \#(nP \cap \mathbb{Z}^2) = (2n)^2$ ... is a polynomial

Thm: (Ehrhart) 1962

Given a d-dimensional lattice polytope. Then $\#(nP \cap \mathbb{Z}^d)$ is a polynomial in n of degree d.

$$L(P,n) := c_0 + c_1 n + c_2 n^2 + \cdots + c_d n^d$$

Question: What do the coefficients c_i mean?

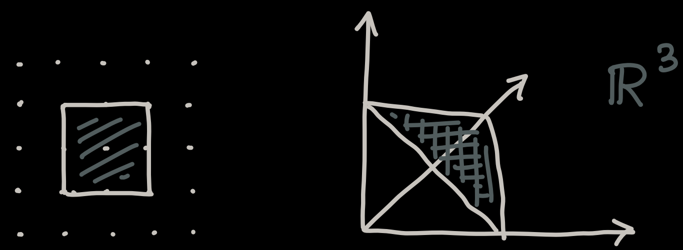
- c_d = volume of P (this generalizes Pick's theorem)
- $c_{d-1} \approx$ surface volume for convex polytopes
- c_0 = Euler characteristic $= 1$

Reciprocity:

- $L(P, -1) = (-1)^d \times$ interior lattice points of P

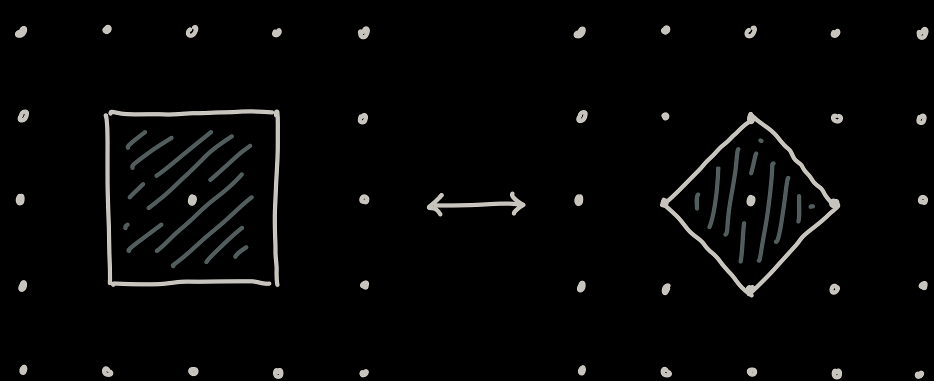
(Ehrhart - Macdonald reciprocity)

Question: which polytopes are lattice polytopes for sufficiently high-dimensional $\mathbb{Z}^d$?



Thm: The regular n-gon for $n \geq 5$ is <u>not</u> a lattice polytope.

Def: a lattice polytope is **reflexive** if its polar dual is a lattice polytope as well.



Thm: For each $d \in \mathbb{N}$ exist only finitely many reflexive d-polytopes.

Other subdirections:

- 01-polytopes (e.g. matroids, enumeration, ...)
- empty & hollow lattice polytopes (finitely many, '91)
- toric varieties

8.2. Zonotopes & hyperplane arrangements

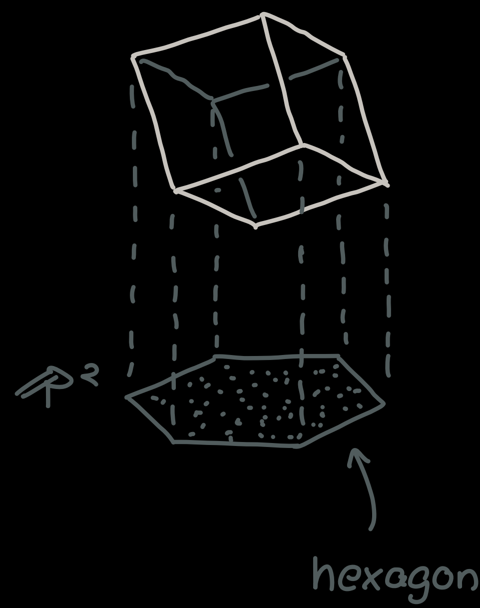
Def: a zonotop $Z \subset \mathbb{R}^d$

(i) is the projection of a cube.

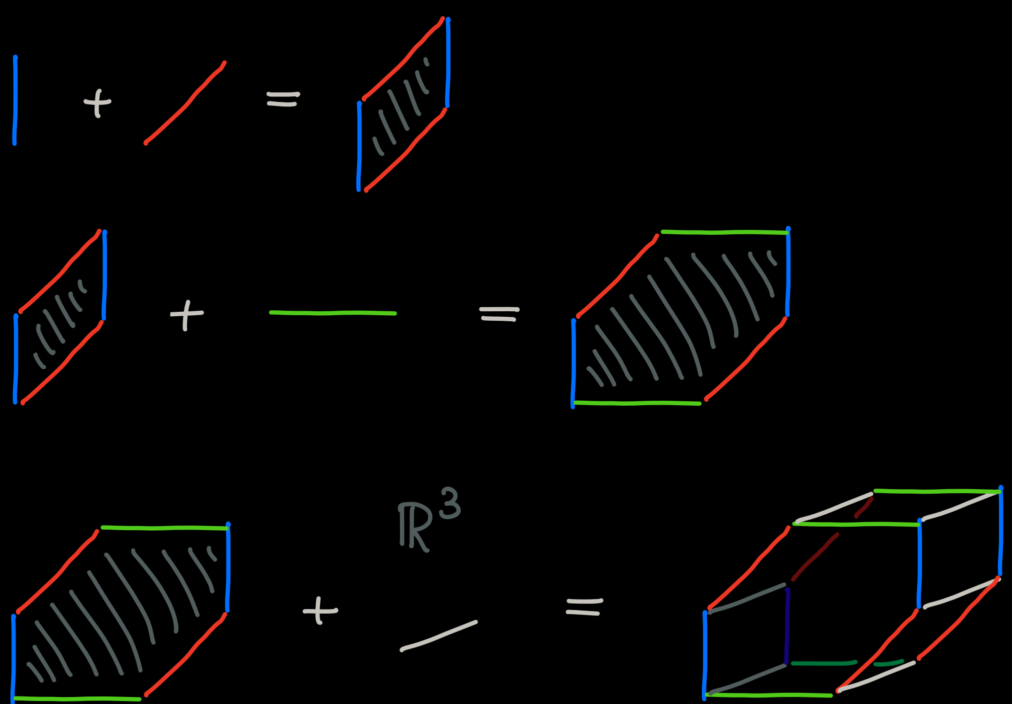
(ii) is the Minkowski sum of line segments.

(iii) has only centrally symmetric faces.

(iv) has only centrally symmetric 2-faces.

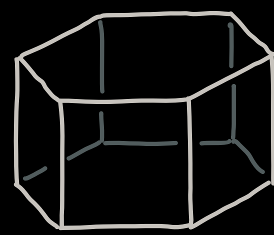


Minkowski sums: $P+Q := \{p+q \mid p \in P, q \in Q\}$



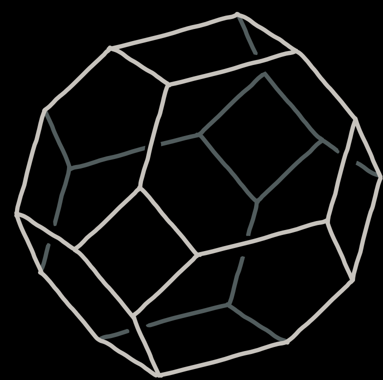
Examples:

1) cubes (in all dimensions)
2) centrally symmetric polygons
3) prisms over the above



4) permutohedron

vertices = permutations of $(1,2,\dots,d+1)$



Generating zonotopes

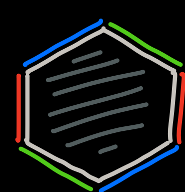
generators

$R \subset \mathbb{R}^d$ finite

$$Z := \mathrm{Zon}(R) := \sum_{r \in R} \mathrm{conv}\{\pm r\}$$

Minkowski sum

$$R = \{ \uparrow, \nearrow, \nwarrow \} \longrightarrow \mathrm{Zon}(R) =$$



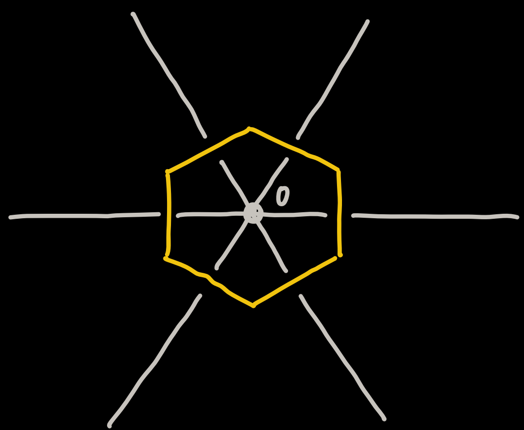
- if no d vectors in $R \subset \mathbb{R}^d$ are linearly dependent, then $\mathrm{Zon}(R)$ is generic.
- generic zonotopes are examples of cubical polytopes
  := all faces are comb. cubes

OPEN: classify the simple 3-zonotopes

(Grünbaum – Cuntz conjecture: 3 families + 90)

Hyperplane arrangements

central hyperplane arrangement $\xleftrightarrow{\text{normal vectors}}$ set of vectors $\xleftrightarrow{\text{generators}}$ zonotopes



vertices of zonotope
=
chambers of hyperplane arrangement

Ex: the combinatorial type of a zonotope is independent of the length of the generators.

Question: Given n (generic) hyperplanes in $\mathbb{R}^d$.
How many chambers do they create?

$\Longleftrightarrow$ how many vertices has a generic d-zonotope with n generators?

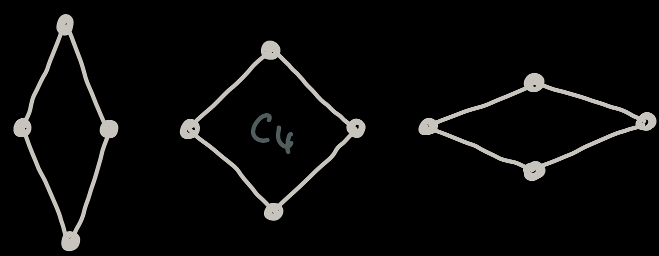
$$f_0 = 2 \sum_{k=0}^{d-1} \binom{n-1}{d-1}$$

8.3. Rigidity theory of polytopes

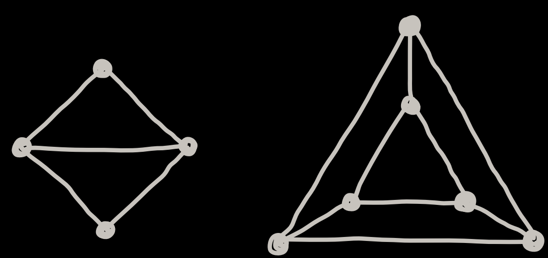
Classical rigidity theory :

- given an embedded graph, can it be deformed continuously with changing edge lengths ?

bar-joint frameworks = embedded graphs



flexible



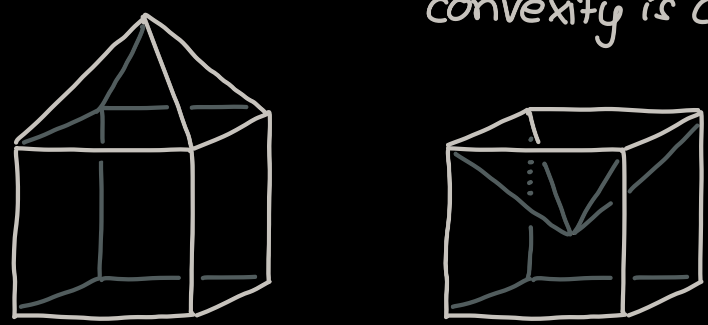
rigid

Thm (Cauchy's rigidity theorem) 1813

(local rigidity)

1) a 3-polytope with rigid 2-faces is itself rigid
2) two 3-polytopes with pair-wise isometric 2-faces are isometric.
3) Let $\varphi : \mathcal{F}(P) \longrightarrow \mathcal{F}(Q)$ be a face lattice isomorphism. If φ extends to an isometry on all $\sigma \in \mathcal{F}_2(P)$, then φ extends to an isometry on all of P.

(global rigidity)



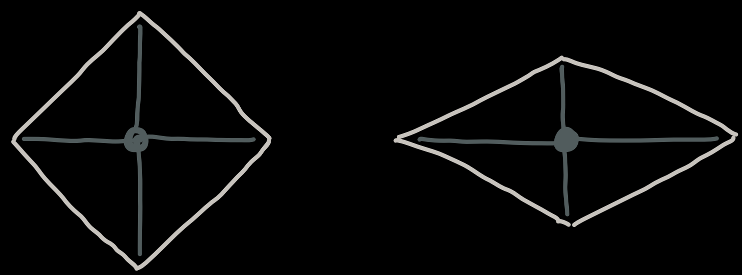
convexity is crucial

NOTES:

- by Cauchy: 1-skeleta of simplicial polytopes are rigid
- <u>not</u> true when surface is non-convex

  → Conelly's flexible polyhedra
- Thm: The 1-skeleton of every simplicial surface, convex or not, that is <u>not</u> a sphere, is always rigid. 2022
- rigidity questions were surprisingly connected to the Upper Bound Conjecture.

OPEN: What if I only know the 2-skeleton + the shape of each 2-face?



OPEN:

- given edge-graph + edge-lengths, is the combinatorial type uniquely determined?
- given edge-graph + edge-lengths + distances of vertices from origin, is the polytope uniquely determined?

  → Yes if

  (i) combinatorial type is known

  (ii) polytope is centrally symmetric

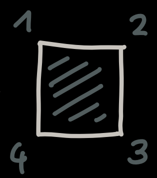
Thm: (Stoker's conjecture, Wang & Xie) 2022

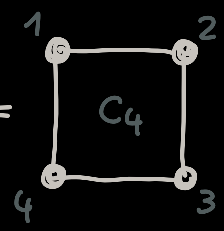
Given a combinatorial type and the dihedral angles, the face angles are uniquely determined.

Thm: (Kalai's conjecture, Novik & Zheng) 2022

The edge-graph + space of self-stresses determin the polytope up to affine equivalence.

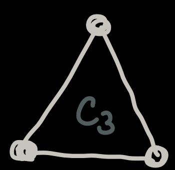
8.4. Spectral polytope theory

$P =$ 

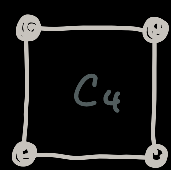
$G_P =$ 

Adjacency matrix

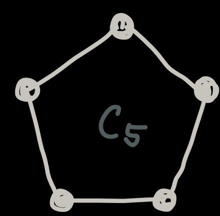
$$A(G_P) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array} \in \mathbb{R}^{4\times 4}$$



$\{-1^{2}, 2^{1}\}$

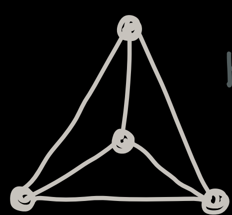


$\{-2^{1}, 0^{2}, 2^{1}\}$



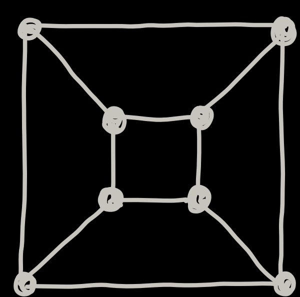
$\{-\varphi^{2}, \psi^{2}, 2^{1}\}$

$\varphi = \frac{\sqrt{5}+1}{2}$, $\psi = \frac{\sqrt{5}-1}{2}$



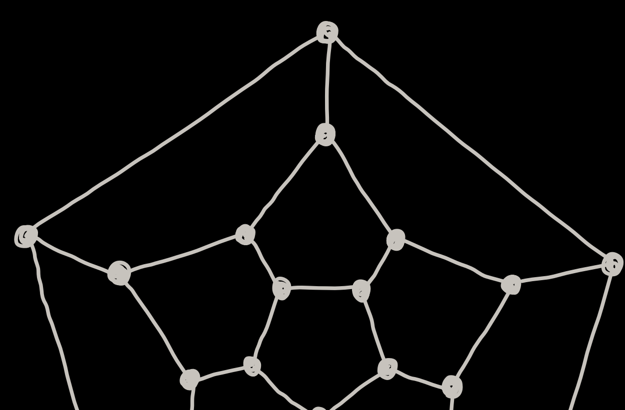
K_4

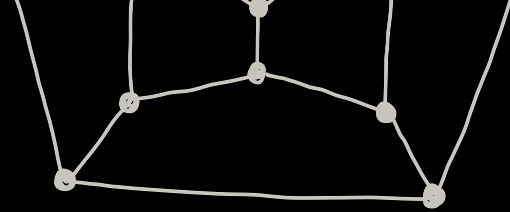
$\{-1^{3}, 3^{1}\}$



$\{-3^{1}, -1^{3}, 1^{3}, 3^{1}\}$

$K_n \quad \{-1^{n-1}, n-1^{1}\}$



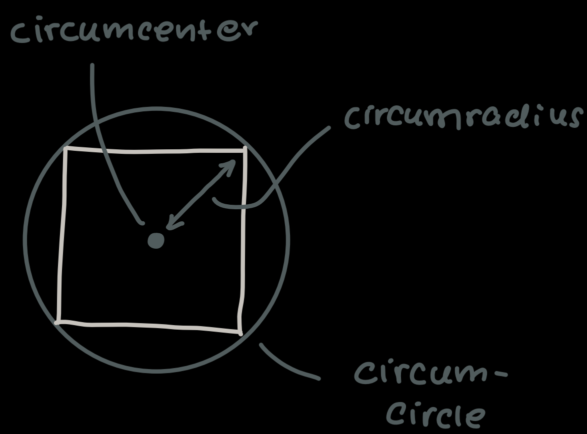


$$\{-\sqrt{5}^{\,3}, -2^{4}, 0^{4}, 1^{5}, \sqrt{5}^{\,3}, 3^{1}\}$$

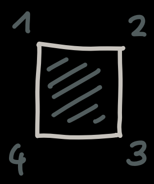
$\theta_2 = \sqrt{5}$

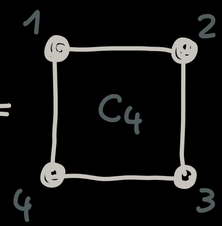
- If edge-lengths are 1, then

$$\text{circumradius} = \left(2 - \frac{2\theta_2}{\deg(G_P)}\right)^{-1/2}$$



Spectral realizations:

$P =$ 

$G_P =$ 

$$A(G_P) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

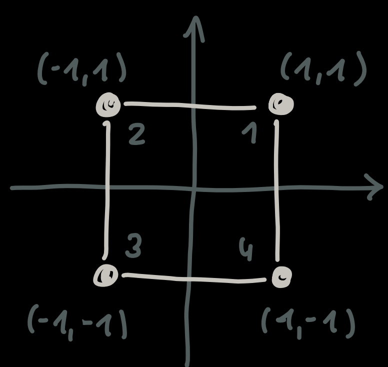
second largest eigenvalue

$$\mathrm{Spec}(A(G_P)) = \{ -2^1, \underset{\theta_2}{0^2}, \underset{\theta_1}{2^1} \}$$

eigenvectors to $\theta_2 = 0$: $(1,-1,1,1) \in \mathbb{R}^4$

$(1,1,-1,-1) \in \mathbb{R}^4$

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

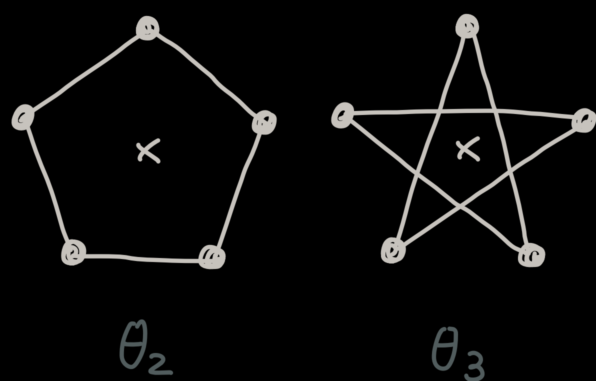


$P_1 = (1,1)$

$P_2 = (-1,1)$

$P_3 = (-1,-1)$

$P_4 = (1,-1)$



Thm: (Izmestiev) 2008

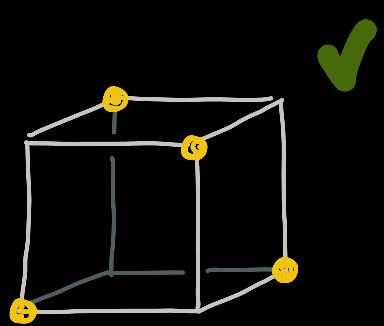
The 1-skeleton of a polytope is a spectral embedding of its (weighted) edge-graph to the second-largest eigenvalue θ_2.

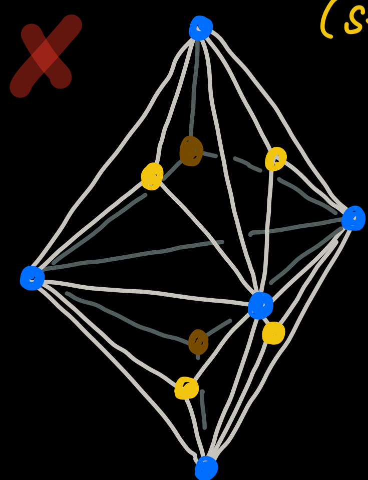
8.5. Inscribed polytopes

Def: $P \subset \mathbb{R}^d$ is *inscribed* (in a sphere) if there exists a sphere $S \subset \mathbb{R}^d$ that contains all vertices of P.

Thm: (Steinitz) 1928

If G has an independent set of size $> \frac{1}{2}|V(G)|$, then a polytope with edge-graph G cannot be inscribed.





(stacked) triangular bipyramid

$5 + 6 = 11$

Thm: (Rivin) ~ 1980

A combinatorial type $\mathcal{F}$ of a 3-polytope is inscribable if there exists a function $\omega: \mathcal{F}_1 \longrightarrow (0, \pi)$ so that

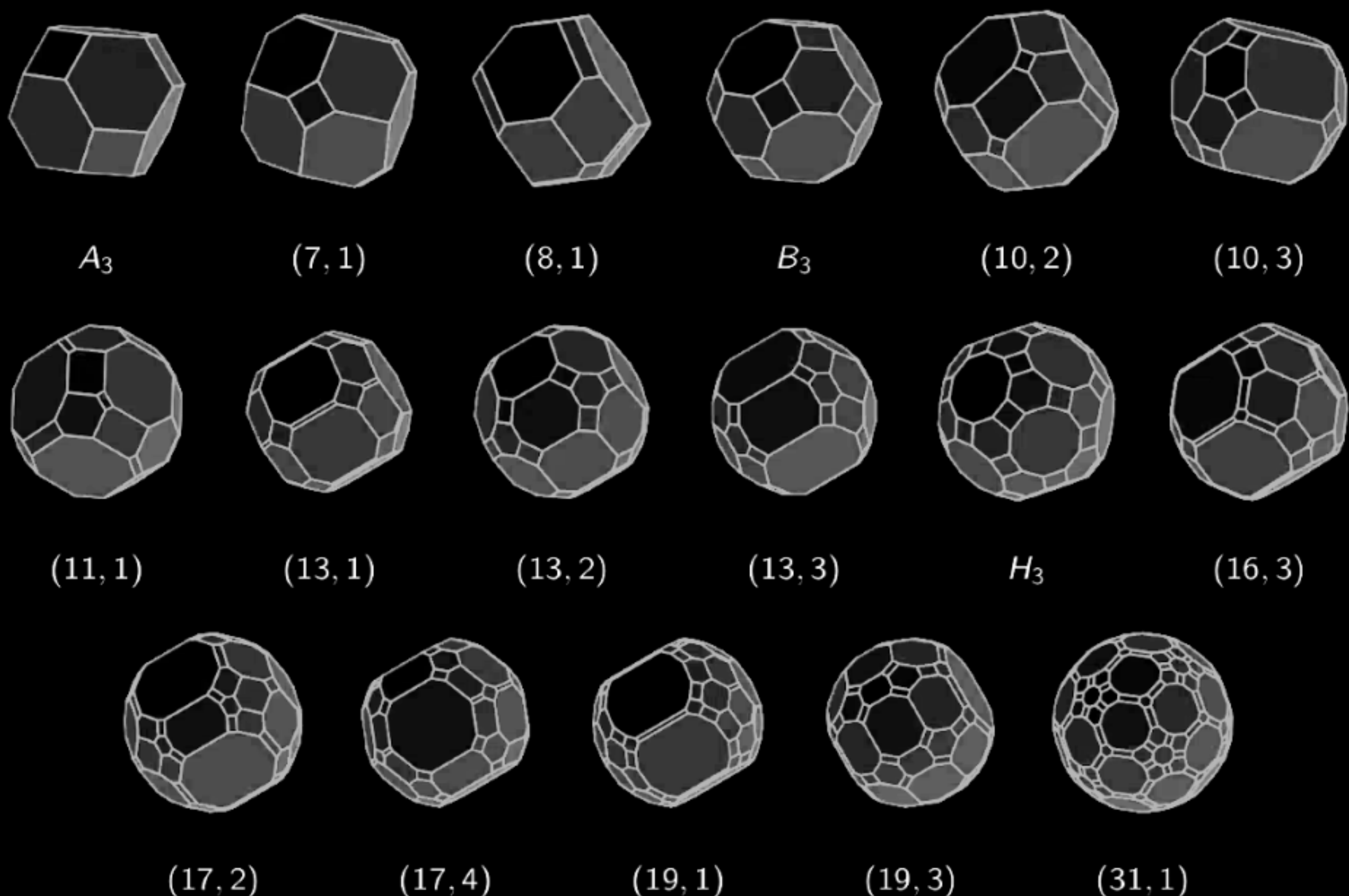
$$\sum_{e \in F} \omega(e) \begin{cases} = 2\pi & F \text{ is edge set of a facet} \\ > 2\pi & F \text{ is edge set of any other simple edge cycle} \end{cases}$$

OPEN: Is there a completely combinatorial description of edge-graphs of inscribed 3-polytopes?

- polytopes with a regular bipartite edge-graph are the boundary case of Steinitz' theorem.
- if a zonotope is inscribed, then it is simple

OPEN: Is the following list 17 inscribed 3-zonotopes complete?

[Manecke & Sanyal]



8.6. Symmetric polytopes

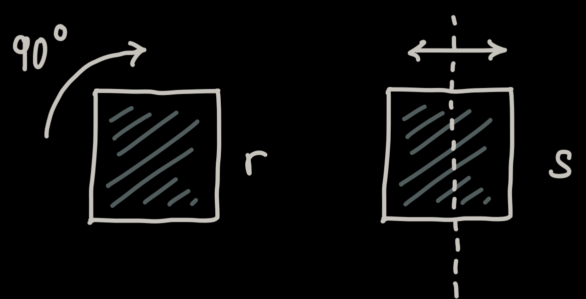
tetrahedron cube octahedron icosahedron dodecahedron

d	2	3	4	≥ 5
#	∞	5	6	3

(Schläfli)

Def: The symmetry group of $P \subset \mathbb{R}^d$ is the group of isometries of $\mathbb{R}^d$ that fix P set-wise

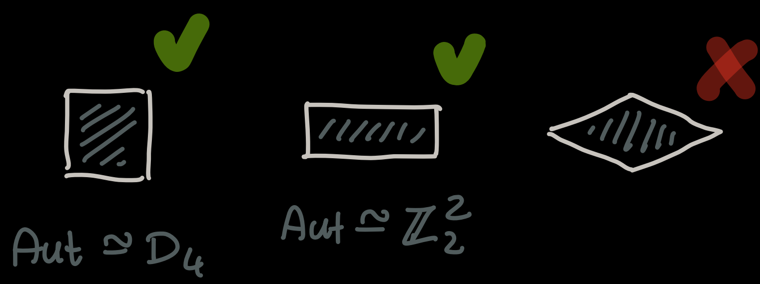
$$\mathrm{Aut}(P) := \{ \text{isometry } T \mid TP = P \}$$



$$\mathrm{Aut}(\square) = \langle r, s \rangle$$

Thm: Every finite group is the symmetry group of a polytope.

Def: vertex-transitive (orbit polytope) := Aut(P) acts transitively on the vertices of P



Thm: (Babai) ~ 1978

Every finite group is the symmetry group of a vertex-transitive polytope except for

(i) abelian groups excluding $\mathbb{Z}_2^r$.

(ii) generalized dicyclic groups.

Thm: If $Z \subset \mathbb{R}^d$ is a vertex-transitive zonotope, then Z is a W-permutahedron. 2020

d	2	3	4	5	6	7	8	≥9
#	∞	3	5	3	4	4	4	3

(combinatorial types)

Def: edge / facet / δ-face - transitive := analogously

Ex: P is vertex-transitive $\longleftrightarrow$ P° is facet transitive

regular := transitive on faces of all dimensions

d	2	3	4	≥ 5
#	∞	5	6	3

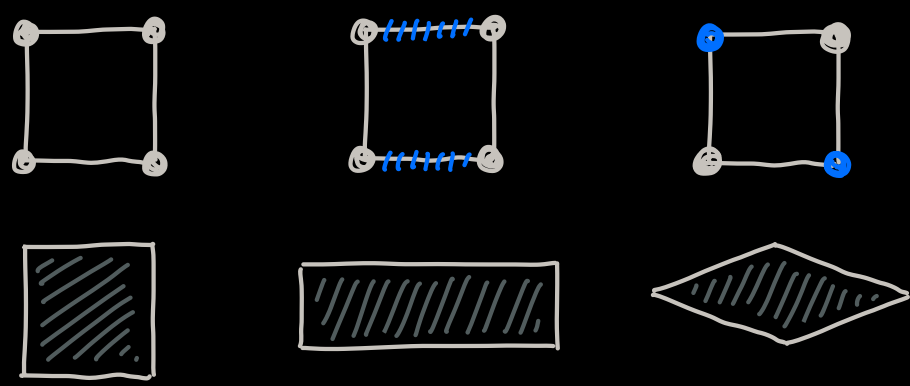
(Schläfli, 18??)

OPEN: classify δ-face-transitive polytopes for some $\delta \in \{1, \dots, d-2\}$ (vertex- and facet-transitive is too wild for classification)

OPEN: Is the following table for the number of edge-transitive d-polytopes correct?

d	2	3	4	5	6	7	8	≥ 9
#	∞	9	15	11	19	22	25	$2d+1$

Thm: There exists a coloring of the edge-graph G_P so that $\mathrm{Aut}(G_P) \simeq \mathrm{Aut}(P)$ 2021



OPEN: 1) are edge colors sufficient if $d \geq 3$?

2) can one color with edge-length and vertex-origin distances ?

8.7. Stanley-Reisner rings

Def: • $\Bbbk[x_1, \dots, x_n]$ … polynomial ring in $\underline{n}$ variables over some field $\Bbbk$

#vertices of P

• face ideal or Stanley-Reisner ideal

$$I_P := \left\langle x_{i_1} \cdots x_{i_r} \;\middle|\; \{i_1, \dots, i_r\} \text{ are } \underline{\text{not}} \text{ a face of } P \right\rangle$$

• face ring or Stanley-Reisner ring

$$\Bbbk[P] := \Bbbk[x_1, \dots, x_n] / I_P$$

• $\Bbbk[P]$ is graded : $\Bbbk[P] = \bigoplus_{\delta} \Bbbk[P]_\delta$

Thm: The (coarse) Hilbert series satisfies

$$H(\Bbbk[P], t) := \sum_{\delta \geq 0} \dim(\Bbbk[P]_\delta)\, t^\delta$$

$$= \frac{1}{(1-t)^n} \sum_{n=0}^{d} f_{i-1}\, t^i (1-t)^{n-i}$$

$$= \frac{h_0 + h_1 t + \dots + h_d t^d}{(1-t)^d}$$

Historical note:

The Upper Bound Conjecture was proven by using algebraic techniques on this ring (Cohen-Macauley property + Reisner's criterion)

8.8. The polytope algebra

Recall: an algebra is a vector space with a multiplication

- matrices + matrix multiplication
- polynomials + usual multiplication
- $\mathbb{R}^3$ + cross product

associative algebras

Def: polytope algebra (of polytopes in $\mathbb{R}^d$)

$$\Pi_d := \mathbb{R}\langle [P] \mid P \subset \mathbb{R}^d \text{ a polytope} \rangle \,/\, \text{Relations}$$

Relations:

Minkowski addition

(i) $[P] \cdot [Q] = [P+Q]$

(ii) $[P \cup Q] + [P \cap Q] = [P] + [Q]$

(iii) $[P+t] = [P]$ for all $t \in \mathbb{R}^d$

One can define exp and log for polytopes:

$$\exp([P]) := \sum_n \frac{[P]^n}{n!}$$

$$[P]^n = [nP]$$

$$\log(\bullet + [P]) := \sum_{n \geq 1} (-1)^{n+1} \frac{[P]^n}{n}$$

The polytope algebra is the universal translation-invariant valuation on polytopes.

$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B)$

e.g. volume, surface area, Euler characteristik, mixed volumes ...